ACADEMIC PRESS

Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Sound and Vibration 270 (2004) 997–1011

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

# Forced transverse vibrations of an elastically connected complex rectangular simply supported double-plate system

Z. Oniszczuk

*Faculty of Mechanical Engineering and Aeronautics, Rzeszów University of Technology, ul. W. Pola 2, 35-959 Rzeszów, Poland*

Received 6 November 2002; accepted 6 February 2003

**Abstract**

The paper deals with forced transverse vibrations of an elastically connected rectangular double-plate system. This complex continuous system can represent a certain simplified model of a three-layered structure consisting of two parallel thin plates separated by an elastic massless layer of a Winkler type. Undamped motion of the system excited by arbitrarily distributed continuous loadings subjected transversely to both plates are governed by a linear set of two coupled non-homogeneous partial differential equations, based on the Kirchhoff–Love plate theory. The forced vibration problem is solved generally by the application of the modal expansion method in the case of simply supported boundary conditions for plates. On the basis of general solutions obtained, three particular cases of the action of exciting stationary harmonic loads are considered. An analysis of harmonic responses of the system makes it possible to determine conditions of resonance and dynamic vibration absorption. A numerical example is given to illustrate the theory presented.
© 2003 Elsevier Ltd. All rights reserved.

## 1. Introduction

Interest in plate theory already has a long and interesting history. Vibrations of elastic plates have been studied extensively for over two centuries [1,2]. Dynamical problems of plates are some of the most important in vibration theory because of various wide engineering applications of plate-type structures in many branches of modern aerospace, aircraft, shipbuilding, civil and mechanical engineering. As is known, a plate constitutes a two-dimensional continuous system for which theoretical vibration analysis in a general case for arbitrary boundary conditions is difficult and complicated. Most vibration monographs devoted to distributed systems contain fundamental theory concerning transverse vibrations of a single rectangular plate [2–28]. As the simplest theoretical sample, the case of a simply supported plate is usually considered.

0022-460X/$ - see front matter © 2003 Elsevier Ltd. All rights reserved.
doi:10.1016/S0022-460X(03)00769-7

An elastically connected double-plate system being an interesting model of complex plate systems is created by coupling two parallel plates by means of an elastic continuous layer. In Refs. [29,30] the author has analyzed undamped free transverse vibrations of a rectangular double-plate system interconnected by a homogeneous Winkler elastic layer and governed by simply supported boundary conditions. In the present work, which is an extension of the above-mentioned paper [30], forced vibrations of this system are investigated.

The dynamics of complex plate-type structures is still a subject of great interest to a number of authors. Different aspects of forced responses for rectangular and circular double-plate (or multi-plate) systems are considered by McElman [31], Sinitsyn [32], Rabinovich et al. [33], Snowdon [34], Filippov et al. [35], Lu et al. [36], Chonan [37–39], Mogilevskii [40], Kokhmanyuk et al. [41], Korenev and Rabinovich [42,43], Oniszczuk [29,44–46], Aida et al. [47], Lueschen and Bergman [48], Arpaci [49], and Szcześniak [50,51]. In the investigation of the title system the papers by Oniszczuk [29,52–55] and Nizioł [56], dealing with transverse vibrations of an analogous double-membrane system can also be helpful, because of the application of the same mathematical methods of solution. Refs. [29,34,47,49] devoted to applying a double-plate system as a dynamic vibration absorber are important, considering its practical significance for suppressing effectively excessive forced harmonic vibrations of corresponding mechanical systems.

In this publication, exact theoretical solutions describing undamped forced transverse vibrations of a rectangular simply supported double-plate system subjected to arbitrarily distributed exciting loadings are formulated. These solutions are next used to determine dynamical responses of plates due to three particular cases of stationary harmonic loads.

## 2. Formulation of the problem

The transverse vibration problem of the title system is formulated in Ref. [30]. The model of the vibratory system shown in Fig. 1 is composed of two parallel rectangular plates joined by an elastic layer. For simplicity of consideration and further analysis, it is assumed that a connecting layer is modelled as the simplest Winkler massless foundation [57]. Both plates are thin, uniform, homogeneous and isotropic. The plates are subjected to arbitrarily distributed transverse continuous loadings and are governed by simply supported boundary conditions. Vibrations of the system with no damping are investigated.

The transverse vibrations of an elastically connected double-plate system are described by the following set of two coupled non-homogeneous partial differential equations [29,30], based on the

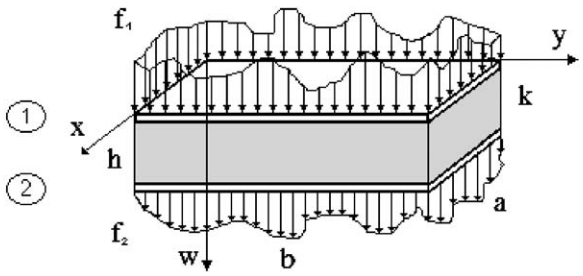


Fig. 1. The general model of an elastically connected complex rectangular simply supported double-plate system.

Kirchhoff–Love plate theory:

$$D_1\Delta^2 w_1 + m_1\ddot{w}_1 + k(w_1 - w_2) = f_1(x,y,t),$$
$$D_2\Delta^2 w_2 + m_2\ddot{w}_2 + k(w_2 - w_1) = f_2(x,y,t), \quad (1)$$

where $w_i = w_i(x,y,t)$ is the transverse plate displacement, $f_i = f_i(x,y,t)$ is the exciting distributed load, $x$, $y$, $t$ are the space co-ordinates and the time, $a$, $b$, $h_i$ are the plate dimensions, $h$, $k$ are the thickness and the stiffness modulus of a Winkler elastic layer respectively, $D_i$ is the flexural rigidity of the plate, $E_i$ is the Young's modulus of elasticity, $v_i$ is the Poisson ratio, $\rho_i$ is the mass density, $D_i = E_i h_i^3[12(1-v_i^2)]^{-1}$, $m_i = \rho_i h_i$, $\dot{w}_i = \partial w_i/\partial t$, $\Delta^2 w_i = \partial^4 w_i/\partial x^4 + 2\partial^4 w_i/\partial x^2\partial y^2 + \partial^4 w_i/\partial y^4$, $i = 1,2$.

The boundary conditions for simply supported plates are:

$$w_i(0,y,t) = w_i(a,y,t) = w_i(x,0,t) = w_i(x,b,t) = 0, \quad i = 1,2,$$
$$\partial^2 w_i/\partial x^2|_{(0,y,t)} = \partial^2 w_i/\partial x^2|_{(a,y,t)} = \partial^2 w_i/\partial y^2|_{(x,0,t)} = \partial^2 w_i/\partial y^2|_{(x,b,t)} = 0. \quad (2)$$

The initial conditions for the problem are assumed in a homogeneous form

$$w_i(x,y,0) = 0, \quad \dot{w}_i(x,y,0) = 0, \quad i = 1,2. \quad (3)$$

## 3. Solution of the forced vibration problem

The vibrations of plates are excited by transverse continuous loads $f_i = f_i(x,y,t)$, $(i = 1,2)$ being arbitrary functions of the space co-ordinates $x,y$ and the time $t$, distributed on a whole surface of both plates. In such a general case of loading, the most proper and useful method of solution of the problem is the modal expansion method [7,29]. Applying the mentioned method, particular solutions of non-homogeneous differential equations (1) representing the forced responses of a double-plate system can be assumed in the form of the following series:

$$w_1(x,y,t) = \sum_{m,n=1}^{\infty}\sum_{i=1}^{2} W_{1imn}(x,y)P_{imn}(t) = \sum_{m,n=1}^{\infty} W_{mn}(x,y)\sum_{i=1}^{2} P_{imn}(t)$$
$$= \sum_{m,n=1}^{\infty} \sin(a_m x)\sin(b_n y)\sum_{i=1}^{2} P_{imn}(t), \quad (4)$$

$$w_2(x,y,t) = \sum_{m,n=1}^{\infty}\sum_{i=1}^{2} W_{2imn}(x,y)P_{imn}(t) = \sum_{m,n=1}^{\infty} W_{mn}(x,y)\sum_{i=1}^{2} a_{imn}P_{imn}(t)$$
$$= \sum_{m,n=1}^{\infty} \sin(a_m x)\sin(b_n y)\sum_{i=1}^{2} a_{imn}P_{imn}(t),$$

where

$$a_{imn} = k^{-1}(D_1 k_{mn}^4 + k - m_1\omega_{imn}^2) = k(D_2 k_{mn}^4 + k - m_2\omega_{imn}^2)^{-1}$$
$$= \omega_{10}^{-2}(\omega_{11mn}^2 - \omega_{imn}^2) = \omega_{20}^2(\omega_{22mn}^2 - \omega_{imn}^2)^{-1}, \quad a_{1mn} > 0, \quad a_{2mn} < 0, \quad (5)$$

$$a_m = a^{-1}m\pi, \quad b_n = b^{-1}n\pi, \quad k_{mn}^4 = (a_m^2 + b_n^2)^2 = \pi^4[(a^{-1}m)^2 + (b^{-1}n)^2]^2,$$
$$\omega_{i0}^2 = km_i^{-1} = KM_i^{-1}, \quad \omega_{iimn}^2 = (D_i k_{mn}^4 + k)m_i^{-1} = (abD_i k_{mn}^4 + K)M_i^{-1},$$

$$\omega_{120}^4 = \omega_{10}^2\omega_{20}^2 = k^2(m_1m_2)^{-1}, \quad K = abk, \quad M_i = abm_i = abh_i\rho_i,$$
$$i = 1,2, \ m,n = 1,2,3,\ldots,$$

$$\omega_{1,2mn}^2 = \tfrac{1}{2}\{(\omega_{11mn}^2 + \omega_{22mn}^2) \mp [(\omega_{11mn}^2 - \omega_{22mn}^2)^2 + 4\omega_{120}^4]\}^{1/2}, \omega_{1mn} < \omega_{2mn}, \tag{6}$$

$$\omega_{1,2mn}^2 = \tfrac{1}{2}\{[(D_1k_{mn}^4 + k)m_1^{-1} + (D_2k_{mn}^4 + k)m_2^{-1}] \mp ([(D_1k_{mn}^4 + k)m_1^{-1} + (D_2k_{mn}^4 + k)m_2^{-1}]^2 - 4k_{mn}^4(m_1m_2)^{-1}[D_1D_2k_{mn}^4 + k(D_1 + D_2)])^{1/2}\},$$

$$W_{1imn}(x,y) = W_{mn}(x,y) = \sin(a_m x)\sin(b_n y),$$
$$W_{2imn}(x,y) = a_{imn}W_{mn}(x,y) = a_{imn}\sin(a_m x)\sin(b_n y), \tag{7}$$

$W_{mn}(x,y)$ (7) are the known natural mode shapes of vibration for a simply supported single plate and $P_{imn}(t)$ are the unknown time functions corresponding to the natural frequencies $\omega_{imn}$. All quantities mentioned above are defined in Ref. [30], where the free vibration problem of the title system is considered. It is important to note that functions $W_{mn}(x,y)$ satisfy the modal equation [7,29]

$$\Delta^2 W_{mn} = k_{mn}^4 W_{mn}. \tag{8}$$

Substituting solutions (4) into the governing equations (1) results in the following relationships:

$$\sum_{m,n=1}^{\infty}\left\{W_{mn}\sum_{i=1}^{2}[m_1\ddot{P}_{imn} + k(1 - a_{imn})P_{imn}] + D_1\Delta^2 W_{mn}\sum_{i=1}^{2}P_{imn}\right\} = f_1(x,y,t),$$
$$\sum_{m,n=1}^{\infty}\left\{W_{mn}\sum_{i=1}^{2}a_{imn}[m_2\ddot{P}_{imn} + k(1 - a_{imn}^{-1})P_{imn}] + D_2\Delta^2 W_{mn}\sum_{i=1}^{2}a_{imn}P_{imn}\right\} = f_2(x,y,t).$$

Including Eq. (8), these relationships can be transformed to the form

$$\sum_{m,n=1}^{\infty} W_{mn}\sum_{i=1}^{2}[\ddot{P}_{imn} + (\omega_{11mn}^2 - \omega_{10}^2 a_{imn})P_{imn}] = m_1^{-1}f_1(x,y,t),$$
$$\sum_{m,n=1}^{\infty} W_{mn}\sum_{i=1}^{2}[\ddot{P}_{imn} + (\omega_{22mn}^2 - \omega_{20}^2 a_{imn}^{-1})P_{imn}]a_{imn} = m_2^{-1}f_2(x,y,t).$$

Taking now expressions (5) and (6) into consideration it follows that

$$\sum_{m,n=1}^{\infty} W_{mn}\sum_{i=1}^{2}(\ddot{P}_{imn} + \omega_{imn}^2 P_{imn}) = m_1^{-1}f_1,$$
$$\sum_{m,n=1}^{\infty} W_{mn}\sum_{i=1}^{2}(\ddot{P}_{imn} + \omega_{imn}^2 P_{imn})a_{imn} = m_2^{-1}f_2.$$

Multiplying both sides of the above equations by the eigenfunction $W_{kl}$ then integrating them over the plate surface and using the corresponding orthogonality condition [29,30]

$$\int_0^a \int_0^b W_{kl} W_{mn}\,\mathrm{d}x\,\mathrm{d}y = \int_0^a \sin(a_k x)\sin(a_m x)\,\mathrm{d}x \int_0^b \sin(b_l y)\sin(b_n y)\,\mathrm{d}y = c\delta_{klmn},$$
$$c = c_{mn}^2 = \int_0^a \int_0^b W_{mn}^2\,\mathrm{d}x\,\mathrm{d}y = \int_0^a \sin^2(a_m x)\,\mathrm{d}x \int_0^b \sin^2(b_n y)\,\mathrm{d}y = 0.25ab, \tag{9}$$

where $\delta_{klmn}$ is the Kronecker delta function: $\delta_{klmn} = 0$ for $k \neq m$ or $l \neq n$, and $\delta_{klmn} = 1$ for $k = m$ and $l = n$; gives

$$\sum_{i=1}^{2} (\ddot{P}_{imn} + \omega_{imn}^2 P_{imn}) = (cm_1)^{-1} \int_0^a \int_0^b f_1 W_{mn}\,\mathrm{d}x\,\mathrm{d}y,$$
$$\sum_{i=1}^{2} (\ddot{P}_{imn} + \omega_{imn}^2 P_{imn}) a_{imn} = (cm_2)^{-1} \int_0^a \int_0^b f_2 W_{mn}\,\mathrm{d}x\,\mathrm{d}y.$$

After some manipulation the following two independent infinite sequences of second order ordinary differential equations for the unknown time functions are found

$$\ddot{P}_{imn} + \omega_{imn}^2 P_{imn} = K_{imn}(t), \quad i = 1, 2, \; m, n = 1, 2, 3, \ldots, \tag{10}$$

where

$$K_{1mn}(t) = d_{1mn} \int_0^a \int_0^b [a_{2mn} M_1^{-1} f_1(x, y, t) - M_2^{-1} f_2(x, y, t)] \sin(a_m x)\sin(b_n y)\,\mathrm{d}x\,\mathrm{d}y,$$
$$K_{2mn}(t) = d_{2mn} \int_0^a \int_0^b [a_{1mn} M_1^{-1} f_1(x, y, t) - M_2^{-1} f_2(x, y, t)] \sin(a_m x)\sin(b_n y)\,\mathrm{d}x\,\mathrm{d}y, \tag{11}$$

$$d_{1mn} = -d_{2mn} = 4(a_{2mn} - a_{1mn})^{-1} = 4\omega_{10}^2(\omega_{1mn}^2 - \omega_{2mn}^2)^{-1}.$$

Their particular solutions satisfying homogeneous initial conditions (3) are as follows [7,24,29,54]

$$P_{imn}(t) = \omega_{imn}^{-1} \int_0^t K_{imn}(s) \sin[\omega_{imn}(t - s)]\,\mathrm{d}s, \quad i = 1, 2, \; m, n = 1, 2, 3, \ldots \;. \tag{12}$$

Finally, the expressions describing the forced vibrations of an elastically connected double-plate system have the following form

$$w_1(x, y, t) = \sum_{m,n=1}^{\infty} \sin(a_m x)\sin(b_n y) \sum_{i=1}^{2} \omega_{imn}^{-1} \int_0^t K_{imn}(s)\sin[\omega_{imn}(t - s)]\,\mathrm{d}s,$$

$$w_2(x, y, t) = \sum_{m,n=1}^{\infty} \sin(a_m x)\sin(b_n y) \sum_{i=1}^{2} a_{imn}\omega_{imn}^{-1} \int_0^t K_{imn}(s)\sin[\omega_{imn}(t - s)]\,\mathrm{d}s. \tag{13}$$

The above-mentioned formulae have a versatile nature and can be used to find the dynamic responses of this system for practically any type of exciting stationary and moving non-inertial transversal loading. For simplicity of further consideration it is assumed that only the first plate is subjected to an arbitrarily distributed continuous load applied on its whole surface (see Fig. 2),

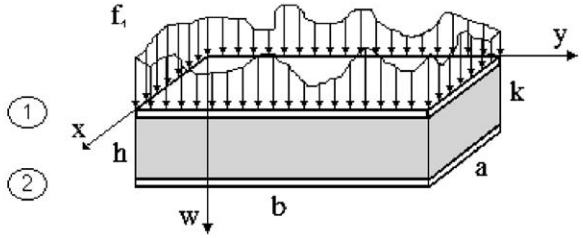


Fig. 2. An elastically connected double-plate system subjected to arbitrarily distributed continuous load.

whilst the other one is not loaded; i.e., $f_1(x,y,t)\neq 0,\ f_2(x,y,t)=0$ . Thus the time functions (11) take the simpler form

$$K_{imn}(t) = c_{imn}\int_0^a\int_0^b f_1(x,y,t)\sin(a_m x)\sin(b_n y)\,\mathrm{d}x\,\mathrm{d}y, \quad i=1,2, \tag{14}$$

where

$$c_{1mn} = a_{2mn}d_{1mn}M_1^{-1} = 4a_{2mn}[(a_{2mn}-a_{1mn})M_1]^{-1},$$
$$c_{2mn} = a_{1mn}d_{2mn}M_1^{-1} = 4a_{1mn}[(a_{1mn}-a_{2mn})M_1]^{-1}.$$

The final form of solution (12) is then as follows

$$P_{imn}(t) = c_{imn}\omega_{imn}^{-1}\int_0^t\left[\int_0^a\int_0^b f_1(x,y,s)\sin(a_m x)\sin(b_n y)\,\mathrm{d}x\,\mathrm{d}y\right]\sin[\omega_{imn}(t-s)]\,\mathrm{d}s. \tag{15}$$

In vibration analysis the most important cases involving forced vibrations are those caused by harmonic forces. Assuming then that the exciting loading is a harmonic function of time $f_1(x,y,t)=f(x,y)\sin(pt)$, relations (14) can be transformed to

$$K_{imn}(t) = c_{imn}\sin(pt)\int_0^a\int_0^b f(x,y)\sin(a_m x)\sin(b_n y)\,\mathrm{d}x\,\mathrm{d}y = c_{imn}F_{mn}\sin(pt), \tag{16}$$

where

$$F_{mn} = \int_0^a\int_0^b f(x,y)\sin(a_m x)\sin(b_n y)\,\mathrm{d}x\,\mathrm{d}y, \quad i=1,2, \tag{17}$$

$f(x,y)$ is the arbitrary function of spatial co-ordinates $x,\ y$, and $p$ is the frequency of harmonic excitation. It is seen that expression (15) can be integrated effectively with respect to time, and as a result one gets

$$\begin{aligned}P_{imn}(t) &= c_{imn}\omega_{imn}^{-1}\int_0^a\int_0^b f(x,y)\sin(a_m x)\sin(b_n y)\,\mathrm{d}x\,\mathrm{d}y\\ &\quad\times\int_0^t \sin(ps)\sin[\omega_{imn}(t-s)]\,\mathrm{d}s\\ &= c_{imn}F_{mn}(\omega_{imn}^2-p^2)^{-1}[\sin(pt)-p\omega_{imn}^{-1}\sin(\omega_{imn}t)].\end{aligned} \tag{18}$$

Taking Eq. (18) into account, the forced harmonic vibrations of system (13) can be presented in the following form

$$w_1(x,y,t) = \sum_{m,n=1}^{\infty} \sin(a_m x)\sin(b_n y)\left[A_{1mn}\sin(pt) + \sum_{i=1}^{2} B_{imn}\sin(\omega_{imn}t)\right],$$

$$w_2(x,y,t) = \sum_{m,n=1}^{\infty} \sin(a_m x)\sin(b_n y)\left[A_{2mn}\sin(pt) + \sum_{i=1}^{2} a_{imn}B_{imn}\sin(\omega_{imn}t)\right], \quad (19)$$

where

$$A_{1mn} = 4F_{mn}M_1^{-1}(\omega_{22mn}^2 - p^2)[(\omega_{1mn}^2 - p^2)(\omega_{2mn}^2 - p^2)]^{-1},$$
$$A_{2mn} = 4F_{mn}M_1^{-1}\omega_{20}^2[(\omega_{1mn}^2 - p^2)(\omega_{2mn}^2 - p^2)]^{-1},$$
$$B_{1mn} = 4a_{2mn}F_{mn}M_1^{-1}p[(a_{1mn} - a_{2mn})\omega_{1mn}(\omega_{1mn}^2 - p^2)]^{-1},$$
$$B_{2mn} = 4a_{1mn}F_{mn}M_1^{-1}p[(a_{2mn} - a_{1mn})\omega_{2mn}(\omega_{2mn}^2 - p^2)]^{-1}. \quad (20)$$

The solutions obtained (19) are composed of two parts. The first part being a function of $\sin(pt)$ denotes the steady state forced vibrations of the system, and the other one containing the terms $\sin(\omega_{imn}t)$ represents the free vibration produced by the application of exciting loading. Neglecting the free response, and assuming that only the steady state response has a practical significance, the forced vibrations of an elastically connected double-plate system are found to be in the form

$$w_1(x,y,t) = \sin(pt)\sum_{m,n=1}^{\infty} A_{1mn}\sin(a_m x)\sin(b_n y),$$

$$w_2(x,y,t) = \sin(pt)\sum_{m,n=1}^{\infty} A_{2mn}\sin(a_m x)\sin(b_n y). \quad (21)$$

Analyzing the steady state vibration amplitudes $A_{imn}$ $(i = 1,2)$ (20) gives the following basic conditions important from a double-plate system dynamics point of view

(a) *condition of resonance*: $p = \omega_{imn}, \quad i = 1,2, \; m,n = 1,2,3,\ldots,$ (22)

(b) *condition of dynamic vibration absorption*:

$$p^2 = p_{mn}^2 = \omega_{22mn}^2 = (D_2k_{mn}^4 + k)m_2^{-1} = (abD_2k_{mn}^4 + K)M_2^{-1}, \; m,n = 1,2,3,\ldots, \quad (23)$$

$$A_{1mn} = 0, \quad A_{2mn} = -4F_{mn}K^{-1} = -4K^{-1}\int_0^a\int_0^b f(x,y)\sin(a_m x)\sin(b_n y)\,dx\,dy. \quad (24)$$

It is proper to add that the above-mentioned conditions are valid for an arbitrary stationary exciting harmonic loading. The resonance phenomenon (22) takes place when the excitation frequency of harmonic load $p$ is equal to that of the double sequence of natural frequencies of the system $\omega_{imn}$ $(i = 1,2, \; m,n = 1,2,3,\ldots)$. The condition (23) implies the phenomenon of dynamic vibration absorption, which occurs, when some physical parameters characterizing the system, namely, the elastic layer stiffness modulus $k$, the flexural rigidity $D_2$, and the mass $M_2$ of the second plate are suitably chosen. Then according to Eq. (24) the amplitude $A_{1mn}$ of any selected harmonic component of the first plate vibrations vanishes, while the amplitude $A_{2mn}$ of the second plate attains a certain finite value. As is seen, in an elastically connected double-plate system, when the first plate (main plate) is subjected to an exciting harmonic load, the second plate can act like a dynamic vibration absorber in relation to the first one. Optimum values of tuning parameters of a dynamic absorber calculated from the basic condition (23) $k, D_2, M_2$ can be then applied to design of continuous dynamic vibration absorbers (CDVAs) of plate type [29,47]. However it should be noted that the continuous absorber only reduces the forced vibrations of the first plate but never liquidates them absolutely.

In order to illustrate the theory developed three simple cases of exciting stationary harmonic loadings are now considered in detail. A detailed vibration analysis is performed for the uniform continuous load distributed on the whole surface of the first plate, for the uniform continuous load distributed along a straight line on the first plate, and for the concentrated force applied transversely at the arbitrary point of the first plate.

*Case* 1: Harmonic uniform distributed surface load.

It is assumed that the harmonic uniform distributed continuous load acts on the whole surface of the first plate $f_1(x,y,t) = f\sin(pt)$, where $f$ and $p$ are the amplitude and frequency of the exciting load respectively (see Fig. 3). Performing the integration in expression (17) gives

$$F_{mn} = f\int_0^a \sin(a_m x)\,\mathrm{d}x \int_0^b \sin(b_n y)\,\mathrm{d}y = 4F(mn\pi^2)^{-1}, \tag{25}$$

where $F = abf,\ m,n = 1,3,5,\dots$ .

Including Eq. (25) in the general relations (19), the system forced vibrations are obtained in the modified form

$$\begin{aligned} w_1(x,y,t) &= \sum_{(m,n)} \sin(a_m x)\sin(b_n y)\left[A_{1mn}\sin(pt) + \sum_{i=1}^{2} B_{imn}\sin(\omega_{imn}t)\right],\\ w_2(x,y,t) &= \sum_{(m,n)} \sin(a_m x)\sin(b_n y)\left[A_{2mn}\sin(pt) + \sum_{i=1}^{2} a_{imn}B_{imn}\sin(\omega_{imn}t)\right], \end{aligned} \tag{26}$$

where

$$\begin{aligned} A_{1mn} &= 16F(M_1 mn\pi^2)^{-1}(\omega_{22mn}^2 - p^2)[(\omega_{1mn}^2 - p^2)(\omega_{2mn}^2 - p^2)]^{-1},\\ A_{2mn} &= 16F(M_1 mn\pi^2)^{-1}\omega_{20}^2[(\omega_{1mn}^2 - p^2)(\omega_{2mn}^2 - p^2)]^{-1},\\ B_{1mn} &= 16Fa_{2mn}(M_1 mn\pi^2)^{-1}p[(a_{1mn} - a_{2mn})\omega_{1mn}(\omega_{1mn}^2 - p^2)]^{-1},\\ B_{2mn} &= 16Fa_{1mn}(M_1 mn\pi^2)^{-1}p[(a_{2mn} - a_{1mn})\omega_{2mn}(\omega_{2mn}^2 - p^2)]^{-1}, \end{aligned} \tag{27}$$

$$i = 1,2,\ \ m,n = 1,3,5,\dots\ .$$

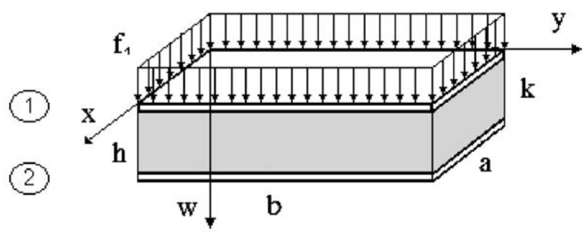


Fig. 3. An elastically connected double-plate system subjected to harmonic uniform distributed surface load.

The steady state forced vibrations of the system are as follows

$$w_1(x,y,t) = \sin(pt) \sum_{(m,n)} A_{1mn} \sin(a_m x)\sin(b_n y),$$
$$w_2(x,y,t) = \sin(pt) \sum_{(m,n)} A_{2mn} \sin(a_m x)\sin(b_n y), \quad (28)$$

$$m,n = 1,3,5,\ldots\ .$$

The analysis of the steady state vibration amplitudes $A_{imn}$ $(i = 1,2)$ (27) makes it possible to formulate the following fundamental conditions:

(a) *condition of resonance*: $p = \omega_{imn},\ i = 1,2,\ m,n = 1,3,5,\ldots,$
(b) *condition of dynamic vibration absorption*:

$$p^2 = p_{mn}^2 = \omega_{22mn}^2 = (abD_2k_{mn}^4 + K)M_2^{-1},$$
$$A_{1mn} = 0, \quad A_{2mn} = -16F(Kmn\pi^2)^{-1}, \quad m,n = 1,3,5,\ldots \quad (29)$$

*Case* 2: Harmonic uniform distributed line load.

The first plate is subjected to harmonic uniform load $f_1(x,y,t) = f\sin(pt)\delta(x - x_0)$ distributed from $y = 0$ to $y = b$ along a straight line e.g., $x = x_0$ parallel to $y$-axis (see Fig. 4), where $\delta(x)$ is the Dirac delta function. Performing the integration in expression (17) it follows that

$$F_{mn} = f\int_0^a \sin(a_m x)\delta(x - x_0)\,\mathrm{d}x \int_0^b \sin(b_n y)\,\mathrm{d}y = 2Fc_m(n\pi)^{-1}, \quad (30)$$

where

$$c_m = \sin(a_m x_0) = \sin(a^{-1}m\pi x_0), \quad F = bf, \quad m = 1,2,3,\ldots,\ n = 1,3,5,\ldots\ .$$

The forced vibrations of plates are obtained in analogous general form to this Eq. (26) in *Case* 1

$$w_1(x,y,t) = \sum_{(m,n)} \sin(a_m x)\sin(b_n y)\left[A_{1mn}\sin(pt) + \sum_{i=1}^{2} B_{imn}\sin(\omega_{imn}t)\right],$$
$$w_2(x,y,t) = \sum_{(m,n)} \sin(a_m x)\sin(b_n y)\left[A_{2mn}\sin(pt) + \sum_{i=1}^{2} a_{imn}B_{imn}\sin(\omega_{imn}t)\right], \quad (31)$$

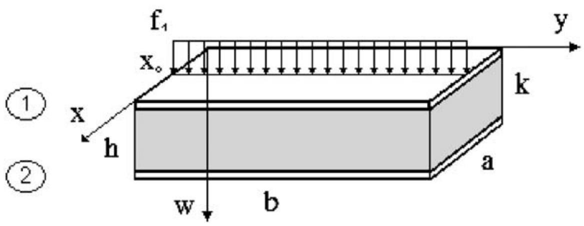


Fig. 4. An elastically connected double-plate system subjected to harmonic uniform distributed line load.

where

$$A_{1mn} = 8Fc_m(M_1 n\pi)^{-1}(\omega^2_{22mn} - p^2)[(\omega^2_{1mn} - p^2)(\omega^2_{2mn} - p^2)]^{-1},$$
$$A_{2mn} = 8Fc_m(M_1 n\pi)^{-1}\omega^2_{20}[(\omega^2_{1mn} - p^2)(\omega^2_{2mn} - p^2)]^{-1},$$
$$B_{1mn} = 8Fa_{2mn}c_m(M_1 n\pi)^{-1}p[(a_{1mn} - a_{2mn})\omega_{1mn}(\omega^2_{1mn} - p^2)]^{-1},$$
$$B_{2mn} = 8Fa_{1mn}c_m(M_1 n\pi)^{-1}p[(a_{2mn} - a_{1mn})\omega_{2mn}(\omega^2_{2mn} - p^2)]^{-1}, \quad (32)$$

$$i = 1, 2, \quad m = 1, 2, 3, \ldots, \quad n = 1, 3, 5, \ldots \; .$$

The steady state forced responses of the system have the form

$$w_1(x, y, t) = \sin(pt) \sum_{(m,n)} A_{1mn} \sin(a_m x) \sin(b_n y),$$
$$w_2(x, y, t) = \sin(pt) \sum_{(m,n)} A_{2mn} \sin(a_m x) \sin(b_n y). \quad (33)$$

The analysis of the steady state vibration amplitudes $A_{imn}$ $(i = 1, 2)$ (32) leads to the two important conditions:

(a) *condition of resonance*: $p = \omega_{imn}, \; i = 1, 2, \; m = 1, 2, 3, \ldots, \; n = 1, 3, 5, \ldots,$
(b) *condition of dynamic vibration absorption:*

$$p^2 = p^2_{mn} = \omega^2_{22mn} = (abD_2 k^4_{mn} + K)M_2^{-1},$$
$$A_{1mn} = 0, \quad A_{2mn} = -8Fc_m(Kn\pi)^{-1}, \; m = 1, 2, 3, \ldots, \; n = 1, 3, 5, \ldots \; . \quad (34)$$

*Case* 3: Harmonic concentrated force.

The first plate is loaded by the concentrated harmonic force applied transversely at the point which position is described by the corresponding rectangular co-ordinates $x = x_0$ and $y = y_0$ (see Fig. 5). The exciting loading of the system can then be presented in the following form: $f_1(x, y, t) = F\sin(pt)\delta(x - x_0)\delta(y - y_0)$, where $F$ and $p$ are the amplitude and frequency of the harmonic force, respectively. Performing the integration in expression (17) one obtains

$$F_{mn} = F \int_0^a \sin(a_m x)\delta(x - x_0)\,dx \int_0^b \sin(b_n y)\delta(y - y_0)\,dy = Fc_{mn}, \quad (35)$$

where

$$c_{mn} = \sin(a_m x_0)\sin(b_n y_0) = \sin(a^{-1}m\pi x_0)\sin(b^{-1}n\pi y_0), \quad m, n = 1, 2, 3, \ldots \; .$$

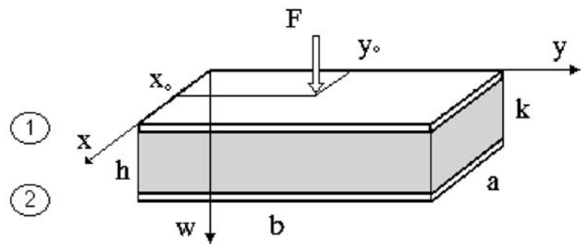


Fig. 5. An elastically connected double-plate system subjected to harmonic concentrated force.

The forced vibrations of plates are obtained in the same general form as (19)

$$w_1(x,y,t) = \sum_{m,n=1}^{\infty} \sin(a_m x)\sin(b_n y)\left[A_{1mn}\sin(pt) + \sum_{i=1}^{2} B_{imn}\sin(\omega_{imn}t)\right],$$

$$w_2(x,y,t) = \sum_{m,n=1}^{\infty} \sin(a_m x)\sin(b_n y)\left[A_{2mn}\sin(pt) + \sum_{i=1}^{2} a_{imn}B_{imn}\sin(\omega_{imn}t)\right], \quad (36)$$

where

$$A_{1mn} = 4Fc_{mn}M_1^{-1}(\omega_{22mn}^2 - p^2)[(\omega_{1mn}^2 - p^2)(\omega_{2mn}^2 - p^2)]^{-1},$$
$$A_{2mn} = 4Fc_{mn}M_1^{-1}\omega_{20}^2[(\omega_{1mn}^2 - p^2)(\omega_{2mn}^2 - p^2)]^{-1},$$
$$B_{1mn} = 4Fa_{2mn}c_{mn}M_1^{-1}p[(a_{1mn} - a_{2mn})\omega_{1mn}(\omega_{1mn}^2 - p^2)]^{-1},$$
$$B_{2mn} = 4Fa_{1mn}c_{mn}M_1^{-1}p[(a_{2mn} - a_{1mn})\omega_{2mn}(\omega_{2mn}^2 - p^2)]^{-1}. \quad (37)$$

The steady state forced vibrations of the system are the following

$$w_1(x,y,t) = \sin(pt)\sum_{m,n=1}^{\infty} A_{1mn}\sin(a_m x)\sin(b_n y),$$

$$w_2(x,y,t) = \sin(pt)\sum_{m,n=1}^{\infty} A_{2mn}\sin(a_m x)\sin(b_n y). \quad (38)$$

The analysis of the steady state vibration amplitudes $A_{imn}$ ($i = 1, 2$) (37) gives the following conditions:

(a) *condition of resonance:* $p = \omega_{imn}, \quad i = 1,2, \ m,n = 1,2,3,\ldots,$
(b) *condition of dynamic vibration absorption:*

$$p^2 = p_{mn}^2 = \omega_{22mn}^2 = (abD_2k_{mn}^4 + K)M_2^{-1},$$
$$A_{1mn} = 0, \quad A_{2mn} = -4Fc_{mn}K^{-1}, \ m,n = 1,2,3,\ldots \ . \quad (39)$$

## 4. Numerical example

As an example illustrating the theory derived, the forced vibrations of a double-plate system consisting of identical plates subjected to harmonic concentrated force $f_1(x,y,t) = F\sin(pt) \times\delta(x - \frac{a}{2})\delta(y - \frac{b}{2})$ applied for simplicity at the middle of the first plate (see *Case* 3 and Fig. 5) are considered in detail. The free vibrations of such a system have been analyzed in Ref. [30]. In the numerical calculations, the data characterizing the physical and geometrical properties of the system discussed are assumed as follows [30]:

$$a = 1\ \text{m}, \quad b = 2\ \text{m}, \quad D = D_i = Eh^3[12(1-\nu^2)]^{-1}, \quad E = E_i = 1\times10^{10}\ \text{N/m}^2,$$
$$h = h_i = 1\times10^{-2}\ \text{m}, \quad k = 0.6\times10^5\ \text{N/m}^3, \quad K = abk = 1.2\times10^5\ \text{N/m},$$

$$m_0 = m_i = \rho h = 0.5 \times 10^2\ \mathrm{kg/m^2}, \quad M = M_i = abm_i = 1 \times 10^2\ \mathrm{kg},$$
$$v = v_i = 0.3, \quad \rho = \rho_i = 5 \times 10^3\ \mathrm{kg/m^3}, \quad i = 1, 2.$$

The steady state forced harmonic vibrations of both plates are represented by the modified expressions (38)

$$w_1(x,y,t) = \sin(pt) \sum_{(m,n)} A_{1mn} \sin(a_m x)\sin(b_n y),$$
$$w_2(x,y,t) = \sin(pt) \sum_{(m,n)} A_{2mn} \sin(a_m x)\sin(b_n y), \tag{40}$$

where

$$A_{1mn} = 4Fc_{mn}M^{-1}(\omega^2_{22mn} - p^2)[(\omega^2_{1mn} - p^2)(\omega^2_{2mn} - p^2)]^{-1},$$
$$A_{2mn} = 4Fc_{mn}M^{-1}\omega^2_{20}[(\omega^2_{1mn} - p^2)(\omega^2_{2mn} - p^2)]^{-1},$$

$$a_m = a^{-1}m\pi, \qquad b_n = b^{-1}n\pi, \quad k^4_{mn} = (a^2_m + b^2_n)^2 = \pi^4[(a^{-1}m)^2 + (b^{-1}n)^2]^2,$$

$$c_{mn} = \sin(0.5m\pi)\sin(0.5n\pi) = (-1)^{0.5(m+n)-1}, \quad K = abk, \quad M = abm_0 = abh\rho, \quad m_0 = h\rho,$$

$$\omega^2_{22mn} = (Dk^4_{mn} + k)m_0^{-1} = (abDk^4_{mn} + K)M^{-1}, \quad \omega^2_{20} = km_0^{-1} = KM^{-1},$$
$$\omega^2_{1mn} = Dk^4_{mn}m_0^{-1}, \quad \omega^2_{2mn} = (Dk^4_{mn} + 2k)m_0^{-1} = \omega^2_{1mn} + 2\omega^2_{20}, \quad m,n = 1,3,5,\ldots\ .$$

The central position of the applied concentrated force causes the solutions obtained to be expressed only by the symmetric mode shapes of vibration $W_{mn}(x,y) = \sin(a_m x)\sin(b_n y)$ $(m,n = 1,3,5,\ldots)$ [30]. In Fig. 6 the resonant diagram of steady state harmonic response of system (40) is presented. It shows the progress of component amplitudes $A_{imn}$ $(i = 1,2,\ m,n = 1,3,5,\ldots)$ as a function of the exciting frequency $p$. This graph comprises only the first two resonance curves. The full lines represent the amplitudes of the first plate vibration components $A_{111},\ A_{113}$, and the broken lines describe the amplitudes of the second plate vibration components $A_{211},\ A_{213}$. The resonances take place when the excitation frequency of harmonic force is equal to the one of the natural frequencies of the system $p = \omega_{111},\ \omega_{211},\ \omega_{113},\ \omega_{213}$, then the corresponding amplitudes $A_{111},\ A_{211},\ A_{113},\ A_{213}$ tend to infinity. It is also seen that the dynamic vibration absorption phenomenon occurs when the excitation frequency is the same as the one of the tuned exciting frequencies $p = p_{11},\ p_{13}$. The amplitudes $A_{111},\ A_{113}$ are suppressed and vanish when the amplitudes $A_{211},\ A_{213}$ of the second plate attain a certain finite value.

The tuned frequencies are evaluated from the condition of dynamic vibration absorption (39)

$$p^2 = p^2_{mn} = \omega^2_{22mn} = (abDk^4_{mn} + K)M^{-1} = 0.5(\omega^2_{1mn} + \omega^2_{2mn}),$$

which leads to the following plate amplitudes

$$A_{1mn} = 0, \quad A_{2mn} = -4Fc_{mn}K^{-1},\ m,n = 1,3,5,\ldots\ .$$

The dynamic absorption phenomenon can be used to reduce excessive forced harmonic vibrations of elastically connected double-plate systems.

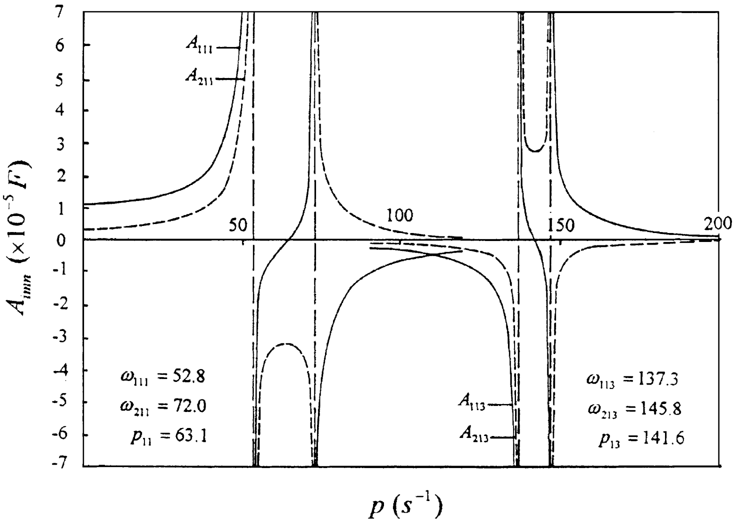


Fig. 6. The resonant diagram of the steady state forced harmonic vibrations of an elastically connected double-plate system subjected to harmonic concentrated force.

## 5. Conclusions

This paper is devoted to the analysis of undamped forced transverse vibrations of an elastically connected rectangular simply supported double-plate system. General solutions of the problem formulated for isotropic, thin plates subjected to arbitrarily distributed continuous loads are found by applying the classical modal expansion method. Three types of exciting stationary harmonic loadings, acting on the first plate, namely, uniform distributed surface load, uniform distributed line load, and concentrated force are considered in detail. Analyzing responses of the system excited by the action of these loads, conditions of resonance and dynamic vibration absorption are determined. Tuning parameters obtained can be applied to optimum design of continuous dynamic vibration absorbers (CDVAs) of plate type. As is well known, dynamic absorbers (DVA) are of great practical importance in engineering applications. Among them CDVAs play a considerable role and many recent papers have been treated with plate-type [29,34,47,49], membrane-type [29,52–56], beam-type [29,59–64], string-type [29,53,58,64], and shell-type [65] continuous absorbers.

## References

[1] G. Jemielita, On the winding paths of the theory of plates, *Scientific Works of Warsaw University of Technology, Civil Engineering* 117 (1991) 1–220 (in Polish).
[2] C. Woźniak (Ed.). *Mechanics of Elastic Plates and Shells*, *Technical Mechanics*, Vol. VIII, Scientific Publishers PWN SA, Warsaw, 2001 (in Polish).
[3] S. Ziemba, *Vibration Analysis*, Vol. II, PWN, Warsaw, 1959 (in Polish).
[4] W. Nowacki, *Dynamics of Elastic Systems*, Chapman & Hall, London, 1963.
[5] R. Solecki, M. Szymkiewicz, *Rod and Surface Systems, Dynamical Calculations*, Arkady, Warsaw, 1964 (in Polish).
[6] I.M. Babakov, *Theory of Vibrations*, Nauka, Moscow, 1965 (in Russian).

[7] S. Kaliski, *Vibrations and Waves in Solids*, IPPT PAN, Warsaw, 1966 (in Polish).
[8] E. Skudrzyk, *Simple and Complex Vibratory Systems*, The Pennsylvania State University Press, University Park, PA, 1968.
[9] A.W. Leissa, Vibration of plates, *NASA SP-160*, 1969.
[10] A.P. Filippov, *Vibrations of Deformable Systems*, Mashinostroenie, Moscow, 1970 (in Russian).
[11] V.L. Biderman, *Applied Theory of Mechanical Vibrations*, Vysshaya Shkola, Moscow, 1972 (in Russian).
[12] L. Fryba, *Vibration of Solids and Structures under Moving Loads*, Academia, Prague, 1972.
[13] W. Nowacki, *Dynamics of Structures*, Arkady, Warsaw, 1972 (in Polish).
[14] S.P. Timoshenko, D.H. Young, W. Weaver Jr., *Vibration Problems in Engineering*, Wiley, New York, 1974.
[15] S. Łukasiewicz, *Concentrated Loadings on Plates, Disks and Shells*, PWN, Warsaw, 1976 (in Polish).
[16] Z. Osiński, *Vibration Theory*, PWN, Warsaw, 1978 (in Polish).
[17] W. Nowacki, *Surface Girders*, PWN, Warsaw, 1979 (in Polish).
[18] V.L. Biderman, *Theory of Mechanical Vibrations*, Vysshaya Shkola, Moscow, 1980 (in Russian).
[19] T. Hop, *Layered Structures*, Arkady, Warsaw, 1980 (in Polish).
[20] W. Soedel, *Vibrations of Shells and Plates*, M. Dekker, New York, 1980.
[21] R.R. Craig Jr., *Structural Dynamics*, Wiley, New York, 1981.
[22] D.J. Gorman, *Free Vibration Analysis of Rectangular Plates*, Elsevier, New York, 1982.
[23] Z. Parszewski, *Vibrations and Dynamics of Machines*, WNT, Warsaw, 1982 (in Polish).
[24] S.S. Rao, *Advanced Theory of Vibration*, Wiley, New York, 1992.
[25] S. Kasprzyk, *Dynamics of Continuous Systems*, AGH Publishers, Cracow, 1994 (in Polish).
[26] C.W. de Silva, *Vibration: Fundamentals and Practice*, CRC Press, Boca Raton, FL, 1999.
[27] J. Awrejcewicz, V.A. Krysko, *Vibrations of Continuous Systems*, WNT, Warsaw, 2000 (in Polish).
[28] W. Szcześniak, *Selected Problems on Dynamics of Plates*, Publishing House of Warsaw University of Technology, Warsaw, 2000 (in Polish).
[29] Z. Oniszczuk, *Vibration Analysis of Compound Continuous Systems with Elastic Constraints*, Publishing House of Rzeszów University of Technology, Rzeszów, 1997 (in Polish).
[30] Z. Oniszczuk, Free transverse vibrations of an elastically connected rectangular simply supported double-plate complex system, *Journal of Sound and Vibration* 236 (2000) 595–608.
[31] J.A. McElman, Flutter of two parallel plates connected by an elastic medium, *American Institute of Aeronautics and Astronautics Journal* 2 (1964) 377–379.
[32] A.P. Sinitsyn, Thermoelastic vibrations of a three-layered plate, *Issledovania po Teorii Sooruzenij, Moscow* XIII (1964) 51–61 (in Russian).
[33] I.M. Rabinovich, A.P. Sinitsyn, O.W. Luzhin, B.M. Terenin, *Calculation of Structures under Impuls Interactions*, ILS, Moscow, 1970 (in Russian).
[34] J.C. Snowdon, Platelike dynamic vibration absorbers, *Transactions of the American Society of Mechanical Engineers, Journal of Engineering for Industry* 97 (1975) 88–93.
[35] A.P. Filippov, C.C. Kokhmanyuk, E.G. Yanyutin, *Deformation of Structure Elements under Impact and Impuls Loads*, Naukova Dumka, Kiev, 1978 (in Russian).
[36] Y.P. Lu, J.W. Killian, G.C. Everstine, Vibrations of three layered damped sandwich plate composites, *Journal of Sound and Vibration* 64 (1979) 63–71.
[37] S. Chonan, Elastically connected Mindlin plates subjected to a moving load, *Journal of Sound and Vibration* 63 (1979) 452–454.
[38] S. Chonan, Resonance frequencies and mode shapes of elastically restrained, prestressed annular plates attached together by flexible cores, *Journal of Sound and Vibration* 67 (1979) 487–500.
[39] S. Chonan, Moving load on initially stressed thick plates attached together by a flexible core, *Ingenieur-Archiv* 48 (1979) 143–154.
[40] M.I. Mogilevskii, Forced vibrations of an elastically connected plate system, *Stroitelnaa Mehanika i Raschet Sooruzenij* 6 (1980) 59–63 (in Russian).
[41] C.C. Kokhmanyuk, E.G. Yanyutin, L.G. Romanenko, *Vibrations of Deformable Systems under Impuls and Moving Interactions*, Naukova Dumka, Kiev, 1980 (in Russian).

[42] B.G. Korenev, I.M. Rabinovich, *Dynamic Calculation of Structures under Special Interactions*, Stroiizdat, Moscow, 1981 (in Russian).
[43] B.G. Korenev, I.M. Rabinovich, *Dynamic Calculation of Buildings and Structures*, Stroiizdat, Moscow, 1984 (in Russian).
[44] Z. Oniszczuk, Vibrations of an elastically connected circular double-plate system, *Scientific Works of Rzeszów University of Technology, Mechanics* 126 (42) (1994) 73–87 (in Polish).
[45] Z. Oniszczuk, Vibrations of an elastically connected rectangular double-plate system, *Proceedings of the IXth Symposium on Dynamics of Structures, Rzeszów-Jawor 96*, *Scientific Works of Rzeszów University of Technology, Mechanics* 151, Vol. II, 1996, pp. 117–124 (in Polish).
[46] Z. Oniszczuk, Damped transverse vibrations of an elastically connected double-plate system, Part II: forced vibrations, *Proceedings of the XXth Symposium "Vibrations in Physical Systems"*, *Poznań-Błażejewko*, 2002, pp. 242–243.
[47] T. Aida, K. Kawazoe, S. Toda, Vibration control of plates by plate-type dynamic vibration absorbers, *Transactions of the American Society of Mechanical Engineers, Journal of Vibration and Acoustics* 117 (1995) 332–338.
[48] G.G.G. Lueschen, L.A. Bergman, Green's function synthesis for sandwiched distributed parameter systems, *Journal of Sound and Vibration* 191 (1996) 613–627.
[49] A. Arpaci, Annular plate dampers attached to continuous systems, *Journal of Sound and Vibration* 191 (1996) 781–793.
[50] W. Szcześniak, Vibrations of sandwich plate under moving loads, *Proceedings of the Vth Polish–Russian Seminar "Theoretical Foundations of Civil Engineering"*, *Moscow*, pp. 122–137 (in Polish).
[51] W. Szcześniak, Vibration of elastic sandwich and elastically connected double-plate system under moving loads, *Scientific Works of Warsaw University of Technology, Civil Engineering* 132 (1998) 153–172 (in Polish).
[52] Z. Oniszczuk, Transverse vibrations of an elastically connected rectangular double-membrane compound system, *Proceedings of the XVIIIth Symposium "Vibrations in Physical Systems"*, *Poznań-Błażejewko*, 1998, pp. 209–210.
[53] Z. Oniszczuk, Dynamic vibration absorption in compound continuous systems of two solids connected by elastic constraints, *Proceedings of the VIIIth Polish Symposium "The Influence of Vibrations upon Environment"*, *Cracow-Janowice*, 1998, pp. 269–274 (in Polish).
[54] Z. Oniszczuk, Transverse vibrations of elastically connected rectangular double-membrane compound system, *Journal of Sound and Vibration* 221 (1999) 235–250.
[55] Z. Oniszczuk, Forced vibration analysis of an elastically connected rectangular double-membrane complex system, *Proceedings of the 4th International Scientific Colloquium CAx Techniques '99*, *Bielefeld*, 1999, pp. 393–400.
[56] J. Nizioł, Damping of vibrations of continuous systems in probabilistic formulation, *Proceedings of the IXth Symposium "The Influence of Vibrations upon Environment", Cracow-Janowice, Scientific Works of Cracow University of Technology, Mechanics* 83 (2001) 213–220 (in Polish).
[57] G. Jemielita, W. Szcześniak, The foundation models, *Scientific Works of Warsaw University of Technology, Civil Engineering* 120 (1993) 5–49 (in Polish).
[58] Z. Oniszczuk, Transverse vibrations of elastically connected double-string complex system, Part II: forced vibrations, *Journal of Sound and Vibration* 232 (2000) 367–386.
[59] Z. Oniszczuk, Forced transverse vibrations of an elastically connected complex simply supported double-beam system, *Journal of Sound and Vibration* 264 (2003) 273–286.
[60] T. Aida, S. Toda, N. Ogawa, Y. Imada, Vibration control of beams by beam-type dynamic vibration absorbers, *Journal of Engineering Mechanics* 118 (1992) 248–258.
[61] Y.-H. Chen, C.-Y. Lin, Structural analysis and optimal design of a dynamic absorbing beam, *Journal of Sound and Vibration* 212 (1998) 759–769.
[62] K. Kawazoe, I. Kono, T. Aida, T. Aso, K. Ebisuda, Beam-type dynamic vibration absorber comprised of free–free beam, *Journal of Engineering Mechanics* 124 (1998) 248–258.
[63] H.V. Vu, A.M. Ordóñez, B.H. Karnopp, Vibration of a double-beam system, *Journal of Sound and Vibration* 229 (2000) 807–822.
[64] Z. Oniszczuk, Dynamic vibration absorption in complex continuous systems, *Machine Dynamics Problems* 24 (2) (2000) 81–94.
[65] T. Aida, T. Aso, K. Nakamoto, K. Kawazoe, Vibration control of shallow shell structures using a shell-type dynamic vibration absorber, *Journal of Sound and Vibration* 218 (1998) 245–267.